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# ON THE DISCRETIZATION OF BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS

OMAR ABOURA

ABSTRACT. In this paper, we are dealing with the approximation of the process  $(X_t, Y_t, Z_t)$  solution to the backward doubly stochastic differential equation (BDSDE)

$$\begin{aligned} X_s &= x + \int_0^s b(X_r) dr + \int_0^s \sigma(X_r) dW_r, \\ Y_s &= \phi(X_T) + \int_s^T f(r, X_r, Y_r, Z_r) dr + \int_s^T g(r, X_r, Y_r, Z_r) d\overleftarrow{B}_r - \int_s^T Z_r dW_r. \end{aligned}$$

After proving the  $L^2$ -regularity of  $Z$ , we use the Euler scheme to discretize  $X$  and the Zhang approach in order to give a discretization scheme of the process  $(Y, Z)$ .

## 1. INTRODUCTION

Since the pioneering work of E. Pardoux and S. Peng [PP92], backward stochastic differential equations (BSDEs) have been intensively studied during the two last decades. Indeed, this notion has been a very useful tool to study problems in many areas, such as mathematical finance, stochastic control, partial differential equations; see e.g. [MY99] where many applications are described. Discretization schemes for BSDEs have been introduced and studied by several authors. The first papers on this topic are that of V.Bally [Ba97] and D.Chevance [Ch97]. In his thesis, Zhang made an interesting contribution which was the starting point of intense study among which the works of B. Bouchard and N.Touzi [BT04], E.Gobet, J.P. Lemor and X. Warin [GLW05],... The notion of BSDE has been generalized by E. Pardoux and S. Peng [PP94] to that of Backward Doubly Stochastic Differential Equation (BDSDE) as follows. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $T$  denote some fixed terminal time which will be used throughout the paper,  $(W_t)_{0 \leq t \leq T}$  and  $(B_t)_{0 \leq t \leq T}$  be two independent standard Brownian motions defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  and with values in  $\mathbb{R}^d$ , and  $\mathbb{R}$  respectively. On this space we will deal with two families of  $\sigma$ -algebras:

$$\mathcal{F}_t := \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{N}, \quad \widehat{\mathcal{F}}_t := \mathcal{F}_{0,t}^W \vee \mathcal{F}_{0,T}^B \vee \mathcal{N}, \quad \mathcal{H}_t = \mathcal{F}_{0,T}^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{N}, \quad (1.1)$$

where  $\mathcal{F}_{t,T}^B := \sigma(B_r - B_t; t \leq r \leq T)$ ,  $\mathcal{F}_{0,t}^W := \sigma(W_r - W_0; 0 \leq r \leq t)$  and  $\mathcal{N}$  denotes the class of  $\mathbb{P}$  null sets. We remark that  $(\widehat{\mathcal{F}}_t)$  is a filtration,  $(\mathcal{H}_t)$  is a decreasing family of  $\sigma$ -algebras, while  $(\mathcal{F}_t)$  is neither increasing nor decreasing. Given an initial condition  $x \in \mathbb{R}^d$ , let  $(X_t)$  be the  $d$ -dimensional diffusion process defined by

$$X_t = x + \int_0^t b(X_r) dr + \int_0^t \sigma(X_r) dW_r. \quad (1.2)$$

Let  $\xi \in L^2(\Omega)$  be an  $\mathbb{R}^d$ -valued,  $\mathcal{F}_T$ -measurable random variable,  $f$  and  $g$  be regular enough coefficients; consider the BDSDE defined as follows:

$$\begin{aligned} Y_s &= \xi + \int_s^T f(r, X_r, Y_r, Z_r) dr \\ &\quad + \int_s^T g(r, X_r, Y_r, Z_r) d\overleftarrow{B}_r - \int_s^T Z_r dW_r. \end{aligned} \quad (1.3)$$

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In this equation,  $dW$  is the forward integral and  $d\overleftarrow{B}$  is the backward integral (we send the reader to [NP88] for more details on backward integration). A solution to (1.3) is a pair of real-valued process  $(Y_t, Z_t)$ , such that  $X_t$  and  $Y_t$  are  $(\mathcal{F}_t)$  for every  $t \in [0, T]$ , such that (1.3) is satisfied and

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} |Y_s|^2\right) + \mathbb{E} \int_0^T |Z_s|^2 ds < +\infty. \quad (1.4)$$

In [PP94] Pardoux and Peng have proved that under some Lipschitz property on  $f$  and  $g$  which will be stated more precisely in section 2, (1.3) has a unique solution  $(Y, Z)$ .

The aim of this paper is to study the discretization of a Backward Doubly Stochastic Differential Equation. For the sake of simplicity, as in Zhang's paper [Z04], we assume that  $Y$  and  $Z$  are real-valued processes. The extension to higher dimension is cumbersome and without theoretical problems. This discretization scheme of  $(Y, Z)$  is motivated by the link between (1.3) and the following backward stochastic partial differential equation when  $\xi = \phi(X_T)$  for a regular function  $\phi$ :

$$\begin{aligned} u(t, x) &= \phi(x) + \int_t^T \left( \mathcal{L}u(s, x) + f(s, x, u(s, x), \nabla u(s, x)\sigma(x)) \right) ds \\ &\quad + \int_t^T g(s, x, u(s, x), \nabla u(s, x)\sigma(x)) d\overleftarrow{B}_s, \end{aligned} \quad (1.5)$$

where  $\mathcal{L}$  is the differential operator defined by:

$$\mathcal{L}u(t, x) = \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^*)_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(t, x) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} u(t, x).$$

The paper is organized as follows: first we prove the  $L^2$ -regularity of  $Z$  in section 2. This is a crucial step in order to the scheme using Zhang's method, which is done in section 3. Finally, a numerical scheme is described in the last section. To ease notations, we set  $\Theta_r := (X_r, Y_r, Z_r)$  for  $r \in [0, T]$ . As usual, we denote by  $C_p$  a constant which depends on some parameter  $p$ , and which can change from one line to the next one. Finally, for some function  $h(t, x, y, z)$  defined on  $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ , we let  $\partial_y h(t, x, y, z)$  (resp.  $\partial_z h(t, x, y, z)$ ) the partial derivatives of  $h$  with respect to the real variable  $y$  (resp.  $z$ ), while  $\partial_x h(t, x, y, z)$  will denote the vector  $(\partial_{x_i} h(t, x, y, z), i = 1, \dots, d)$ .

## 2. REGULARITY PROPERTIES

In this section we give some regularity properties of the process  $X, Y$  and  $Z$ .

The following assumptions which ensure existence and uniqueness of the solution will be in force throughout the paper. For every integer  $n \geq 1$ , let  $M^2([0, T], \mathbb{R}^n)$  denote the set of  $\mathbb{R}^n$ -valued jointly measurable processes  $(\varphi_t, t \in [0, T])$  such that  $\varphi_t$  is  $\mathcal{F}_t$ -measurable for almost every  $t$  and  $\mathbb{E} \int_0^T |\varphi_t|^2 dt < +\infty$ .

**Assumption 1** (for the forward process  $X$ ). *The maps  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  are of class  $\mathcal{C}_b^3$ .*

**Assumption 2** (for the backward process  $(Y, Z)$ ). *Let  $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be such that  $f$  and  $g$  are jointly measurable, for every  $(x, y, z) \in \mathbb{R}^{d+2}$ ,  $f(\cdot, x, y, z)$  and  $g(\cdot, x, y, z)$  belong to  $M^2([0, T], \mathbb{R})$ , and such that:*

- (i) *There exist some nonnegative constants  $L_f, L_g$  and a constant  $\alpha \in [0, 1]$  such that for every  $\omega \in \Omega$ ,  $t, t' \in [0, T]$ ,  $x, x' \in \mathbb{R}^d$ ,  $y, y' \in \mathbb{R}$  and  $z, z' \in \mathbb{R}$*

$$\begin{aligned} |f(t, x, y, z) - f(t', x', y', z')|^2 &\leq L_f \left( |t - t'| + |x - x'|^2 + |y - y'|^2 + |z - z'|^2 \right), \\ |g(t, x, y, z) - g(t', x', y', z')|^2 &\leq L_g \left( |t - t'| + |x - x'|^2 + |y - y'|^2 \right) + \alpha |z - z'|^2, \end{aligned}$$

- (ii) *For all  $s \in [0, T]$   $f(s, \cdot)$  and  $g(s, \cdot)$  are of class  $\mathcal{C}^3$  with bounded partial derivatives up to order 3, uniformly in time.*

(iii) For a function  $h(t, x, y, z)$ , set  $h(t, 0) := h(t, 0, 0, 0)$ . Then

$$\sup_{r \in [0, T]} |f(r, 0)| + \sup_{r \in [0, T]} |g(r, 0)| < \infty.$$

**Assumption 3.** Suppose that  $\xi := \phi(X_T)$  for some function  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  of class  $\mathcal{C}_b^2$  and that for every  $\omega \in \Omega$ ,

$$\sup_{t, x, y, z} |\partial_z g(t, x, y, z)| < 1.$$

**2.1. Some classical properties of the forward process  $X$ .** We at first recall without proof the following well known results on diffusion processes. Define the  $\mathbb{R}^{d \times d}$ -valued process  $(\nabla X_t)_{0 \leq t \leq T}$  by:

$$\nabla X_t := \left( \frac{\partial}{\partial x_j} X_t^i, i, j = 1, \dots, d \right).$$

Then  $\nabla X_t$  is an invertible  $d \times d$  matrix, solution to a linear stochastic differential equation with coefficients depending on  $X_t$ . Furthermore, the assumptions on the coefficients  $\sigma$  and  $b$  yield the following classical result:

**Proposition 2.1.** (i) For all  $p \geq 1$ , there exist a constant  $C_p > 0$  such that for all  $t, s \in [0, T]$ :

$$\mathbb{E} |X_t - X_s|^{2p} + \mathbb{E} \left| (\nabla X_t)^{-1} - (\nabla X_s)^{-1} \right|^{2p} \leq C_p |t - s|^p.$$

(ii) For all  $p \in [1, +\infty[$ , there exist a constant  $C_p > 0$  such that

$$\mathbb{E} \left( \sup_{t \in [0, T]} |X_t|^{2p} + \sup_{t \in [0, T]} \left| (\nabla X_t)^{-1} \right|^p \right) \leq C_p.$$

**2.2. Time increments of  $Y$  and  $L^2$ -regularity of  $Z$ .** The following lemma provides upper bounds for time increments of  $Y$ .

**Lemma 2.2.** Set  $\xi = \phi(X_T)$  for some function  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  be of class  $\mathcal{C}_b^1$ . Then we have

(i) For all  $p \geq 2$ , there exist a constant  $C_p > 0$  depending on  $T$  such that for all  $t, s \in [0, T]$

$$\mathbb{E} |Y_t - Y_s|^p \leq C_p |t - s|^{\frac{p}{2}}. \quad (2.1)$$

(ii) For all  $p \geq 1$ , there exist a constant  $C > 0$  such that

$$\sup_{0 \leq r \leq T} \mathbb{E} |Z_r|^{2p} \leq C. \quad (2.2)$$

Notice that the inequality (2.1) is different from equation (2.11) in [Z04].

*Proof.* We at first prove (ii). Let  $(\nabla Y_t)_{0 \leq t \leq T} = (\partial_x Y_t)_{0 \leq t \leq T}$  denote the real-valued process defined by differentiation of  $Y$  as function of the initial condition  $x$  of the diffusion process  $(X_t)$ . We recall the following representation of  $Z$  (see [PP94] Proposition 2.3):

$$Z_t = \nabla Y_t (\nabla X_t)^{-1} \sigma(X_t). \quad (2.3)$$

where  $(\nabla Y_t, \nabla Z_t)$  satisfies the linear BDSDE with the forward process  $(X_t, \nabla X_t)$  and the evolution equation:

$$\begin{aligned} \nabla Y_t = & \phi'(X_T) \nabla X_T + \int_t^T \left( f_x(r, \Theta_r) \nabla X_r + f_y(r, \Theta_r) \nabla Y_r + f_z(r, \Theta_r) \nabla Z_r \right) dr \\ & + \int_t^T \left( g_x(r, \Theta_r) \nabla X_r + g_y(r, \Theta_r) \nabla Y_r + g_z(r, \Theta_r) \nabla Z_r \right) d\overleftarrow{B}_r - \int_t^T \nabla Z_r dW_r. \end{aligned} \quad (2.4)$$

By E.Pardoux and S.Peng [PP94] page 217, we deduce

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |\nabla Y_t|^p \right) < \infty. \quad (2.5)$$

Then Hölder's inequality and Proposition 2.1 yield

$$\mathbb{E} |Z_t|^{2p} \leq \left( \mathbb{E} |\nabla Y_t|^{6p} \right)^{\frac{1}{3}} \left( \mathbb{E} \left| (\nabla X_t)^{-1} \right|^{6p} \right)^{\frac{1}{3}} \left( \mathbb{E} |\sigma(X_t)|^{6p} \right)^{\frac{1}{3}}.$$

This concludes the proof of (ii).

(i) Suppose that  $s < t$ , then using (1.3), we deduce that

$$\begin{aligned} |Y_t - Y_s|^p &\leq C_p \left| \int_s^t f(r, \Theta_r) - f(r, X_r, Y_r, 0) dr \right|^p + C_p \left| \int_s^t f(r, X_r, Y_r, 0) dr \right|^p \\ &\quad + C_p \left| \int_s^t g(r, \Theta_r) d\overleftarrow{B}_r \right|^p + C_p \left| \int_s^t Z_r dW_r \right|^p. \end{aligned}$$

Recall that  $\widehat{\mathcal{F}}_t$  and  $\mathcal{H}_t$  have been defined in (1.1). The process  $(\int_0^t Z_r dW_r, 0 \leq t \leq T)$  is a  $(\widehat{\mathcal{F}}_t)$ -martingale, while the process  $(\int_t^T g(r, \Theta_r) d\overleftarrow{B}_r, 0 \leq t \leq T)$  is a backward martingale for  $(\mathcal{H}_t)$ . Hence, the Burkholder-Davies-Gundy and Hölder inequalities yield

$$\begin{aligned} \mathbb{E} |Y_t - Y_s|^p &\leq C_p |t - s|^{p-1} \mathbb{E} \int_s^t |f(r, X_r, Y_r, 0)|^p dr + C_p \mathbb{E} \left| \int_s^t Z_r dr \right|^p \\ &\quad + C_p \mathbb{E} \left( \int_s^t |g(r, \Theta_r)|^2 dr \right)^{\frac{p}{2}} + C_p \mathbb{E} \left( \int_s^t |Z_r|^2 dr \right)^{\frac{p}{2}}. \end{aligned} \quad (2.6)$$

Assumption 2 (i) and (ii), Proposition 2.1 and (1.4) yield

$$\begin{aligned} \mathbb{E} \left( \int_s^t |g(r, \Theta_r)|^2 dr \right)^{\frac{p}{2}} &\leq C_p \mathbb{E} \left( \int_s^t |g(r, \Theta_r) - g(r, 0)|^2 dr \right)^{\frac{p}{2}} + C_p \mathbb{E} \left( \int_s^t |g(r, 0)|^2 dr \right)^{\frac{p}{2}} \\ &\leq C_p |t - s|^{\frac{p}{2}} + C_p \mathbb{E} \left( \int_s^t (|X_r|^2 + |Y_r|^2) dr \right)^{\frac{p}{2}} + C_p \mathbb{E} \left( \int_s^t |Z_r|^2 dr \right)^{\frac{p}{2}} \\ &\leq C_p |t - s|^{\frac{p}{2}} + C_p \mathbb{E} \left( \int_s^t |Z_r|^2 dr \right)^{\frac{p}{2}}. \end{aligned} \quad (2.7)$$

Similarly,

$$\begin{aligned} \mathbb{E} \int_s^t |f(r, X_r, Y_r, 0)|^p dr &\leq C_p \mathbb{E} \int_s^t |f(r, 0)|^p dr + C_p \mathbb{E} \int_s^t |f(r, X_r, Y_r, 0) - f(r, 0)|^p dr \\ &\leq C_p \mathbb{E} \int_s^t |f(r, 0)|^p dr + C_p \int_s^t \mathbb{E} (|X_r|^p + |Y_r|^p) dr \leq C_p |t - s|. \end{aligned} \quad (2.8)$$

Hence, the inequalities (2.6)-(2.8) imply

$$\mathbb{E} |Y_t - Y_s|^p \leq C_p |t - s|^{\frac{p}{2}} + C_p \mathbb{E} \left( \int_s^t |Z_r|^2 dr \right)^{\frac{p}{2}}.$$

Using Hölder's inequality and (2.2) we conclude the proof of (2.1).  $\square$

Since equation (2.4) proves that the pair  $(\nabla Y, \nabla Z)$  is the solution of a BDSDE with forward process  $(X, \nabla X) \in L^p$  for every  $p \in [1, +\infty[$ , we deduce from (2.1) that for every function  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  of class  $\mathcal{C}_b^2$ , we have for  $0 \leq s < t \leq T$  and  $p \in [1, +\infty[$ :

$$\mathbb{E} |\nabla Y_t - \nabla Y_s|^p \leq C_p |t - s|^{\frac{p}{2}}, \quad (2.9)$$

for some constant  $C_p > 0$ . We now establish some control of time increments of the process  $Z$ , following the idea of J.Zhang [Z04].

**Theorem 2.3** ( $L^2$ -regularity of  $Z$ ). *There exists a non negative constant  $C$  such that for every subdivision  $\pi = \{t_0 = 0 < t_1 < \dots < t_n = T\}$  with mesh  $|\pi|$ , one has*

$$\sum_{1 \leq i \leq n} \mathbb{E} \int_{t_{i-1}}^{t_i} (|Z_t - Z_{t_{i-1}}|^2 + |Z_t - Z_{t_i}|^2) dt \leq C |\pi|. \quad (2.10)$$

*Proof.* Using the representation of  $Z$  as a product, we deduce (2.3),

$$Z_t - Z_{t_i} = \nabla Y_t (\nabla X_t)^{-1} \sigma(X_t) - \nabla Y_{t_i} (\nabla X_{t_i})^{-1} \sigma(X_{t_i}).$$

Then,

$$\begin{aligned} |Z_t - Z_{t_i}|^2 &\leq 3 |\nabla Y_t - \nabla Y_{t_i}|^2 \left| (\nabla X_t)^{-1} \right|^2 |\sigma(X_t)|^2 \\ &\quad + 3 |\nabla Y_{t_i}|^2 \left| (\nabla X_t)^{-1} - (\nabla X_{t_i})^{-1} \right|^2 |\sigma(X_t)|^2 \\ &\quad + 3 |\nabla Y_{t_i}|^2 \left| (\nabla X_{t_i})^{-1} \right|^2 |\sigma(X_t) - \sigma(X_{t_i})|^2. \end{aligned}$$

To conclude the proof, we use Hölder's inequality, Proposition 2.1 and (2.9).  $\square$

Theorem 2.3 immediatly yields the following

**Corollary 2.4.**

$$\sum_{1 \leq i \leq n-1} \mathbb{E} \int_{t_{i-1}}^{t_i+1} |Z_r - Z_{t_i}|^2 dr \leq C|\pi|.$$

### 3. THE DISCRETIZATION OF $(X, Y, Z)$

**3.1. Discretization of the process  $X$ : The Euler scheme.** We briefly recall the Euler scheme and send the reader to [KP99] for more details. Let  $\pi := \{t_0 = 0 < t_1 < \dots < t_n = T\}$  be a subdivision of  $[0, T]$ . We define the process  $X_t^\pi$ , called the Euler scheme, by

$$X_t^\pi = X_{t_0}^\pi + \int_{t_0}^t b(X_{s_\pi}^\pi) ds + \int_{t_0}^t \sigma(X_{s_\pi}^\pi) dW_s,$$

where  $s_\pi := \max\{t_i \leq s\}$ . The following result is well known:

**Proposition 3.1.** *There exists a constant  $C > 0$  such that for every subdivision  $\pi$ ,*

$$\max_i \mathbb{E} |X_{t_i} - X_{t_i}^\pi|^2 \leq C|\pi|, \quad \mathbb{E} \int_{t_{i-1}}^{t_i} |X_r - X_{t_i}^\pi|^2 dr \leq C|\pi|^2.$$

**3.2. Discretization of the process  $(Y, Z)$ : The step process.** In this section, we construct an approximation of  $(Y, Z)$  using Zhang's approach.

Let  $\pi : t_0 = 0 < \dots < t_n = T$  be any subdivision on  $[0, T]$ . Set  $\mathcal{G}_t = \mathcal{G}_t^i$  for  $t_{i-1} \leq t < t_i$ , where we let

$$\mathcal{G}_t^i := \sigma(W_r - W_0; 0 \leq r \leq t) \vee \sigma(B_r - B_{t_{i-1}}; t_{i-1} \leq r \leq T), \quad t_{i-1} \leq t \leq t_i,$$

and define the  $(\mathcal{G}_t)$ -adapted process  $(Y_t^\pi, Z_t^\pi)_{0 \leq t \leq T}$  recursively (in a backward manner), as follows: Set  $Y_{t_n}^\pi = \phi(X_{t_n}^\pi)$ ,  $Z_{t_n}^{\pi,1} = 0$ ; for  $i = n-1, \dots, 0$ , let

$$Z_{t_i}^{\pi,1} := \frac{1}{\Delta t_{i+1}} \mathbb{E} \left( \int_{t_i}^{t_{i+1}} Z_r^\pi dr \middle| \mathcal{F}_{t_i} \right),$$

and for  $i = n, \dots, 1$ , let

$$\begin{aligned} \Delta t_i &= t_i - t_{i-1}, \Delta B_{t_i} = B_{t_i} - B_{t_{i-1}}, \Theta_{t_i}^{\pi,1} := \left( X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^{\pi,1} \right), \\ Y_t^\pi &= Y_{t_i}^\pi + f(t_i, \Theta_{t_i}^{\pi,1}) \Delta t_i + g(t_i, \Theta_{t_i}^{\pi,1}) \Delta B_{t_i} - \int_t^{t_i} Z_r^\pi dW_r, \quad \forall t \in [t_{i-1}, t_i]. \end{aligned} \quad (3.1)$$

Note that the equation (3.1) is not a BDSDE in the sense of [PP94]; however, we have the following:

**Proposition 3.2.** *For every  $i = 1, \dots, n$ , there exists a process  $(Y_t^\pi, Z_t^\pi)_{t \in [t_{i-1}, t_i]}$  adapted to the filtration  $(\mathcal{G}_t, t_{i-1} \leq t < t_i)$ , such that (3.1) holds. Furthermore,  $Y_{t_i}^\pi \in \mathcal{F}_{t_i}$ .*

*Proof.* The proof is similar to that in [PP94] page 212 and relies on the martingale representation theorem. Fix an integer  $i > 0$  and suppose that the processes  $(Y_t^\pi)$  and  $(Z_t^\pi)$  have been defined for  $t \geq t_i$ ,  $(\mathcal{G}_t)$ -adapted, and that  $Y_{t_k}^\pi$  is  $\mathcal{F}_{t_k}$ -measurable for  $k = i, \dots, n$ . We denote by  $(M_t^i)_{t \in [t_{i-1}, t_i]}$  the process defined by

$$M_t^i := \mathbb{E} \left( Y_{t_i}^\pi + f(t_i, \Theta_{t_i}^{\pi,1}) \Delta t_i + g(t_i, \Theta_{t_i}^{\pi,1}) \Delta B_{t_i} \middle| \mathcal{G}_t^i \right), \quad t_{i-1} \leq t \leq t_i.$$

By the martingale representation theorem, there exists a  $(\mathcal{G}_t^i, t_{i-1} \leq t \leq t_i)$ -adapted and square integrable process  $(N_t^i, t_{i-1} \leq t \leq t_i)$  such that for  $t_{i-1} \leq t \leq t_i$ ,  $M_t^i = M_{t_{i-1}}^i + \int_{t_{i-1}}^t N_s^i dW_s$ . Therefore,  $M_t^i = M_{t_i}^i - \int_t^{t_i} N_s^i dW_s$ . Clearly,  $\mathcal{G}_{t_i}^i$  contains  $\mathcal{F}_{t_i}$ ,  $X_{t_i}^\pi$  is  $\mathcal{F}_{t_i}^W \subset \mathcal{F}_{t_i}$  measurable and  $\Theta_{t_i}^{\pi,1}$  is  $\mathcal{F}_{t_i}$ -measurable; hence

$$M_{t_i}^i = Y_{t_i}^\pi + f\left(t_i, \Theta_{t_i}^{\pi,1}\right) \Delta t_i + g\left(t_i, \Theta_{t_i}^{\pi,1}\right) \Delta B_{t_i}.$$

Furthermore, note that  $\mathcal{G}_{t_{i-1}}^i = \mathcal{F}_{t_{i-1}}$ , so that  $M_{t_{i-1}}^i$  is  $\mathcal{F}_{t_{i-1}}$ -measurable. This completes the proof by setting:  $Y_t^\pi = M_t^i$ ,  $Z_t^\pi = N_t^i$  for  $t_{i-1} \leq t < t_i$ .  $\square$

Before stating the main theorem of this section, we introduce the following

**Definition 3.3.** Let  $\kappa \geq 1$  be a constant. The subdivision  $\pi$  is said to be  $\kappa$ -uniform if  $\kappa \Delta t_i \geq |\pi|$  for every  $i \in \{1, \dots, n\}$ .

The main example of a  $\kappa$ -uniform subdivision is a uniform subdivision (i.e. for all  $i$ ,  $\Delta t_i = |\pi|$ ) where  $\kappa = 1$ . The following lemma gives an upper estimate of  $Z_{t_i} - Z_{t_i}^{\pi,1}$ .

**Lemma 3.4.** For any  $i = 0, \dots, n-1$ , any  $\kappa$ -uniform subdivision  $\pi$  and  $\beta > 0$  we have:

$$\Delta t_i \mathbb{E} \left| Z_{t_i} - Z_{t_i}^{\pi,1} \right|^2 \leq \kappa (1 + \beta) \int_{t_i}^{t_{i+1}} |Z_r^\pi - Z_r|^2 dr + \kappa (1 + \beta^{-1}) \int_{t_i}^{t_{i+1}} |Z_r - Z_{t_i}|^2 dr.$$

*Proof.* For any  $i = 0, \dots, n-1$ ,  $Z_{t_i}$  is  $\mathcal{F}_{t_i}$ -measurable, and  $\Delta t_i \leq |\pi| \leq \kappa \Delta t_{i+1}$ ; thus

$$\begin{aligned} \Delta t_i \mathbb{E} \left| Z_{t_i} - Z_{t_i}^{\pi,1} \right|^2 &= \Delta t_i \mathbb{E} \left| Z_{t_i} - \frac{1}{\Delta t_{i+1}} \mathbb{E} \left( \int_{t_i}^{t_{i+1}} Z_r^\pi dr \middle| \mathcal{F}_{t_i} \right) \right|^2 \\ &= \frac{\Delta t_i}{(\Delta t_{i+1})^2} \mathbb{E} \left| \mathbb{E} \left( \int_{t_i}^{t_{i+1}} (Z_{t_i} - Z_r^\pi) dr \middle| \mathcal{F}_{t_i} \right) \right|^2 \\ &\leq \frac{\kappa}{\Delta t_{i+1}} \mathbb{E} \left| \int_{t_i}^{t_{i+1}} (Z_{t_i} - Z_r^\pi) dr \right|^2 \leq \kappa \mathbb{E} \int_{t_i}^{t_{i+1}} |Z_{t_i} - Z_r^\pi|^2 dr. \end{aligned}$$

where the last step is deduced from Schwarz's inequality. Using the usual estimate  $|Z_{t_i} - Z_r^\pi|^2 \leq (1 + \beta)|Z_r^\pi - Z_r|^2 + (1 + \beta^{-1})|Z_r - Z_{t_i}|^2$ , we conclude the proof.  $\square$

The following theorem is the main result of this section. It proves that as  $|\pi| \rightarrow 0$ ,  $(Y^\pi, Z^\pi)$  converges to  $(Y, Z)$ .

**Theorem 3.5.** Let  $\pi$  be a  $\kappa$ -uniform subdivision with sufficiently small mesh  $|\pi|$ ,  $\alpha < \frac{1}{\kappa}$ , let  $\phi \in \mathcal{C}^2$  and  $\xi = \phi(X_T)$ . Then we have

$$\max_{0 \leq i \leq n} \mathbb{E} |Y_{t_i} - Y_{t_i}^\pi|^2 + \mathbb{E} \int_0^T |Z_r - Z_r^\pi|^2 dr \leq C |\pi|. \quad (3.2)$$

*Proof.* Set  $I_n = \mathbb{E} |\phi(X_T) - \phi(X_T^\pi)|^2$  and for  $i = 1, \dots, n$ , let

$$I_{i-1} := \mathbb{E} \left| Y_{t_{i-1}} - Y_{t_{i-1}}^\pi \right|^2 + \mathbb{E} \int_{t_{i-1}}^{t_i} |Z_r - Z_r^\pi|^2 dr.$$

Using (1.3) with  $\xi = \phi(X_T)$  and (3.1), we deduce

$$\begin{aligned} Y_{t_{i-1}} - Y_{t_{i-1}}^\pi + \int_{t_{i-1}}^{t_i} (Z_r - Z_r^\pi) dW_r &= Y_{t_i} - Y_{t_i}^\pi + \int_{t_{i-1}}^{t_i} \left( f(r, \Theta_r) - f(t_i, \Theta_{t_i}^{\pi,1}) \right) dr \\ &\quad + \int_{t_{i-1}}^{t_i} \left( g(r, \Theta_r) - g(t_i, \Theta_{t_i}^{\pi,1}) \right) d\overleftarrow{B}_r. \end{aligned} \quad (3.3)$$

By construction,  $Y_{t_{i-1}} - Y_{t_{i-1}}^\pi$  is  $\mathcal{F}_{t_{i-1}} = \mathcal{G}_{t_{i-1}}^i$  measurable while for  $r \in [t_{i-1}, t_i)$ ,  $Z_r - Z_r^\pi$  is  $(\mathcal{G}_r)$ -adapted. Hence,  $Y_{t_{i-1}} - Y_{t_{i-1}}^\pi$  is orthogonal to  $\int_{t_{i-1}}^{t_i} (Z_r - Z_r^\pi) dW_r$ . Therefore,

$$I_{i-1} = \mathbb{E} \left| Y_{t_{i-1}} - Y_{t_{i-1}}^\pi + \int_{t_{i-1}}^{t_i} (Z_r - Z_r^\pi) dW_r \right|^2.$$

Since  $g(r, \Theta_r)$  (resp.  $g(t_i, \Theta_{t_i}^{\pi,1})$ ) is  $\mathcal{F}_r$  ( resp.  $\mathcal{F}_{t_i}$ )-measurable, the random variables  $Y_{t_i} - Y_{t_i}^\pi$  and  $\int_{t_{i-1}}^{t_i} (g(r, X_r, Y_r) - g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi)) d\overleftarrow{B}_r$  are orthogonal. Hence for every  $\epsilon > 0$ , using assumption 2, the  $L^2$ -isometry of backward stochastic integrals, Schwarz's inequality and (3.3), we deduce

$$\begin{aligned} I_{i-1} &\leq \left(1 + \frac{\Delta t_i}{\epsilon}\right) \mathbb{E} |Y_{t_i} - Y_{t_i}^\pi|^2 + \left(1 + 2\frac{\epsilon}{\Delta t_i}\right) \mathbb{E} \left| \int_{t_{i-1}}^{t_i} \left(f(r, \Theta_r) - f(t_i, \Theta_{t_i}^{\pi,1})\right) dr \right|^2 \\ &\quad + \left(1 + \frac{\Delta t_i}{\epsilon}\right) \mathbb{E} \left| \int_{t_{i-1}}^{t_i} \left(g(r, \Theta_r) - g(t_i, \Theta_{t_i}^{\pi,1})\right) d\overleftarrow{B}_r \right|^2 \\ &\leq (1 + \Delta t_i \epsilon^{-1}) \mathbb{E} |Y_{t_i} - Y_{t_i}^\pi|^2 + (\Delta t_i + 2\epsilon) \mathbb{E} \int_{t_{i-1}}^{t_i} \left| f(r, \Theta_r) - f(t_i, \Theta_{t_i}^{\pi,1}) \right|^2 dr \\ &\quad + (1 + \Delta t_i \epsilon^{-1}) \mathbb{E} \int_{t_{i-1}}^{t_i} \left| g(r, \Theta_r) - g(t_i, \Theta_{t_i}^{\pi,1}) \right|^2 dr \\ &\leq \left[ 1 + \Delta t_i \epsilon^{-1} + 2L_f (\Delta t_i^2 + 2\epsilon \Delta t_i) + 2L_g (\Delta t_i + \Delta t_i^2 \epsilon^{-1}) \right] \mathbb{E} |Y_{t_i} - Y_{t_i}^\pi|^2 \\ &\quad + \left[ L_f (\Delta t_i + 2\epsilon) + L_g (1 + \Delta t_i \epsilon^{-1}) \right] \mathbb{E} \int_{t_{i-1}}^{t_i} \left( |\pi| + |X_r - X_{t_i}^\pi|^2 + 2|Y_r - Y_{t_i}|^2 \right) dr \\ &\quad + \left[ L_f (\Delta t_i + 2\epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] \mathbb{E} \int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}^{\pi,1}|^2 dr. \end{aligned}$$

For  $|\pi| \leq 1$ ,  $\Delta t_i^2 \leq \Delta t_i$ ; using Proposition 2.1 with  $p = 2$  and Proposition 3.1, we deduce

$$\mathbb{E} \int_{t_{i-1}}^{t_i} \left( |\pi| + |X_r - X_{t_i}^\pi|^2 + 2|Y_r - Y_{t_i}|^2 \right) dr \leq C|\pi|^2,$$

for some constant  $C > 0$ . Hence for any  $\gamma > 0$

$$\begin{aligned} I_{i-1} &\leq \left[ 1 + \left( \epsilon^{-1} + 2L_f(1 + 2\epsilon) + 2L_g(1 + \epsilon^{-1}) \right) \Delta t_i \right] \mathbb{E} |Y_{t_i} - Y_{t_i}^\pi|^2 \\ &\quad + C \left[ L_f (\Delta t_i + \epsilon) + L_g (1 + \Delta t_i \epsilon^{-1}) \right] |\pi|^2 \\ &\quad + (1 + \gamma^{-1}) \left[ L_f (\Delta t_i + 2\epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] \mathbb{E} \int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}|^2 dr \\ &\quad + (1 + \gamma) \left[ L_f (\Delta t_i + 2\epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] \Delta t_i \mathbb{E} |Z_{t_i} - Z_{t_i}^{\pi,1}|^2. \end{aligned}$$



Lemma 3.4 yields for some positive constants  $C_\epsilon$ ,  $C_{\epsilon,\gamma}$  and  $C_{\epsilon,\gamma,\beta}$ , we have:

$$\begin{aligned}
I_{i-1} &\leq (1 + C_\epsilon \Delta t_i) \mathbb{E} |Y_{t_i} - Y_{t_i}^\pi|^2 + C_\epsilon |\pi|^2 + C_{\epsilon,\gamma} \mathbb{E} \int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}|^2 dr \\
&\quad + \kappa (1 + \gamma) (1 + \beta) \left[ L_f (\Delta t_i + 2\epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] \mathbb{E} \int_{t_i}^{t_{i+1}} |Z_r^\pi - Z_r|^2 dr \\
&\quad + \kappa (1 + \gamma) (1 + \beta^{-1}) \left[ L_f (\Delta t_i + 2\epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] \mathbb{E} \int_{t_i}^{t_{i+1}} |Z_r - Z_{t_i}|^2 dr \\
&\leq (1 + C_\epsilon \Delta t_i) \mathbb{E} |Y_{t_i} - Y_{t_i}^\pi|^2 + C_\epsilon |\pi|^2 + C_{\epsilon,\gamma,\beta} \mathbb{E} \int_{t_{i-1}}^{t_{i+1}} |Z_r - Z_{t_i}|^2 dr \\
&\quad + \kappa (1 + \gamma) (1 + \beta) \left[ L_f (\Delta t_i + 2\epsilon) + \alpha (1 + \Delta t_i \epsilon^{-1}) \right] \int_{t_i}^{t_{i+1}} |Z_r^\pi - Z_r|^2 dr. \tag{3.4}
\end{aligned}$$

Recall that  $\alpha < \frac{1}{\kappa}$  and let  $0 < \delta < 1 - \kappa\alpha$ . Then choose positive constants  $\beta$  and  $\gamma$  small enough to ensure  $\kappa (1 + \gamma) (1 + \beta) \alpha < 1 - \frac{2\delta}{3}$ . Finally, let  $\epsilon > 0$  small enough to ensure that  $2\kappa (1 + \gamma) (1 + \beta) L_f \epsilon < \frac{\delta}{6}$ . Then (3.4) implies the existence of  $C > 0$  such that for every  $i = 1, \dots, n-1$ ,

$$I_{i-1} + \frac{\delta}{3} \mathbb{E} \int_{t_i}^{t_{i+1}} |Z_r^\pi - Z_r|^2 dr \leq (1 + C \Delta t_i) I_i + C |\pi|^2 + C \mathbb{E} \int_{t_{i-1}}^{t_{i+1}} |Z_r - Z_{t_i}|^2 dr. \tag{3.5}$$

Using the discrete Gronwall lemma in [Z04] (Lemma 5.4 page 479), we deduce

$$\begin{aligned}
\max_{0 \leq i \leq n} I_i &\leq C e^{CT} \mathbb{E} \left( I_n + \sum_{1 \leq i \leq n-1} \int_{t_{i-1}}^{t_{i+1}} |Z_r - Z_{t_i}|^2 dr + |\pi| \right) \\
&\leq C \mathbb{E} \left( |\phi(X_T) - \phi(X_T^\pi)|^2 + \sum_{1 \leq i \leq n} \int_{t_{i-1}}^{t_i} \left( |Z_r - Z_{t_{i-1}}|^2 + |Z_r - Z_{t_i}|^2 \right) dr + |\pi| \right).
\end{aligned}$$

Since  $\phi$  is Lipschitz, Proposition 3.1 implies that  $\mathbb{E} |\phi(X_T) - \phi(X_T^\pi)|^2 \leq C |\pi|$ ; thus Theorem 2.3 implies

$$\max_{0 \leq i \leq n} \mathbb{E} |Y_{t_i} - Y_{t_i}^\pi|^2 \leq C |\pi|. \tag{3.6}$$

Moreover, summing both sides of (3.5) over  $i$  from 1 to  $n-1$  and using Corollary 2.4 we obtain:

$$\begin{aligned}
\sum_{0 \leq i \leq n-2} I_i + \frac{\delta}{3} \mathbb{E} \int_{t_1}^T |Z_r^\pi - Z_r|^2 dr &\leq \sum_{1 \leq i < n} (1 + C \Delta t_i) I_i + C |\pi| \\
&\quad + C \sum_{1 \leq i < n} \mathbb{E} \int_{t_{i-1}}^{t_{i+1}} |Z_r - Z_{t_i}|^2 dr, \\
&\leq C |\pi| + \sum_{1 \leq i \leq n-1} (1 + C \Delta t_i) I_i.
\end{aligned}$$

Therefore,

$$I_0 + \frac{\delta}{3} \mathbb{E} \int_{t_1}^T |Z_r^\pi - Z_r|^2 dr \leq C |\pi| + I_{n-1} + C \sum_{1 \leq i \leq n-1} \Delta t_i I_i$$

Since  $\delta < 1 - \kappa\alpha < 3$ , using (3.6) we deduce

$$\frac{\delta}{3} \mathbb{E} \int_0^T |Z_r^\pi - Z_r|^2 dr \leq C |\pi| + \mathbb{E} \int_{t_{n-1}}^{t_n} |Z_r^\pi - Z_r|^2 dr + C |\pi| \mathbb{E} \int_0^T |Z_r^\pi - Z_r|^2 dr. \tag{3.7}$$

The equations (1.3) and (3.1) imply

$$\begin{aligned} \int_{t_{n-1}}^{t_n} (Z_r^\pi - Z_r) dW_r &= (Y_{t_n}^\pi - Y_{t_n}) - (Y_{t_{n-1}}^\pi - Y_{t_{n-1}}) \\ &+ \int_{t_{n-1}}^{t_n} (f(t_n, X_{t_n}^\pi, Y_{t_n}^\pi, 0) - f(r, X_r, Y_r, Z_r)) dr \\ &+ \int_{t_{n-1}}^{t_n} (g(t_n, X_{t_n}^\pi, Y_{t_n}^\pi, 0) - g(r, X_r, Y_r, Z_r)) d\bar{B}_r. \end{aligned}$$

The  $L^2$ -isometry, Schwarz's inequality, (3.6), Lemma 2.2, Propositions 2.1 and 3.1

$$\begin{aligned} \mathbb{E} \int_{t_{n-1}}^{t_n} |Z_r^\pi - Z_r|^2 dr &\leq 4\mathbb{E} |Y_{t_n}^\pi - Y_{t_n}|^2 + 4\mathbb{E} |Y_{t_{n-1}}^\pi - Y_{t_{n-1}}|^2 \\ &+ 4|\pi| \mathbb{E} \int_{t_{n-1}}^{t_n} |f(t_n, X_{t_n}^\pi, Y_{t_n}^\pi, 0) - f(r, X_r, Y_r, Z_r)|^2 dr \\ &+ 4\mathbb{E} \int_{t_{n-1}}^{t_n} |g(t_n, X_{t_n}^\pi, Y_{t_n}^\pi, 0) - g(r, X_r, Y_r, Z_r)|^2 dr \\ &\leq C|\pi| + C|\pi| \sup_{t_{n-1} \leq r \leq t_n} \mathbb{E} (|X_r - X_T|^2 + |X_{t_n}^\pi - X_T|^2) \\ &+ C|\pi| \sup_{t_{n-1} \leq r \leq t_n} \mathbb{E} (|Y_r - Y_T|^2 + |Y_{t_n}^\pi - Y_T|^2 + |Z_r|^2) \\ &\leq C|\pi|. \end{aligned} \tag{3.8}$$

For  $|\pi|$  small enough, we have  $C|\pi| \leq \delta/6$ ; thus (3.7) and (3.8) conclude the proof.  $\square$

#### 4. A NUMERICAL SCHEME

In this section we propose a numerical scheme based on the results of the previous sections. First of all, given  $x \in \mathbb{R}^d$ ,  $s < t$  we set:

$$X_t(s, x) := x + (t - s)b(x) + \sigma(x)(W_t - W_s).$$

We clearly have  $X_{t_i}^\pi = X_{t_i}(t_{i-1}, X_{t_{i-1}}^\pi)$  for every  $i = 1, \dots, n$ . Then, given a vector  $(x_0, \dots, x_i; x_{i+1}, \dots, x_n) \in \mathbb{R}^{(i+1)d} \times \mathbb{R}^{n-i}$ , set  $\mathbf{x}_{n+1} = \emptyset$  and for  $i = 0, \dots, n-1$ , let

$$\mathbf{x}^i := (x_0, \dots, x_i), \quad \mathbf{x}_{i+1} := (x_{i+1}, \dots, x_n).$$

Define by induction, the functions  $u_i^\pi, v_i^\pi : \mathbb{R}^{(i+1)d} \times \mathbb{R}^{n-i} \rightarrow \mathbb{R}$  (resp. the random variables  $U_i^\pi, V_i^\pi : \mathbb{R}^{(i+1)d} \times \Omega \times \mathbb{R}^{n-i-1} \rightarrow \mathbb{R}$ ) as follows:

$$u_n^\pi(x_0, \dots, x_n) := \phi(x_n), \quad v_n^\pi(x_0, \dots, x_n) := 0,$$

and for  $i = 0, \dots, n-1$  let

$$U_i^\pi(\mathbf{x}^i, \omega, \mathbf{x}_{i+2}) := u_{i+1}^\pi(\mathbf{x}^i, X_{t_{i+1}}(t_i, x_i), \mathbf{x}_{i+2}) \tag{4.1}$$

$$\begin{aligned} &+ f(t_{i+1}, X_{t_{i+1}}(t_i, x_i), u_{i+1}^\pi(\mathbf{x}^i, X_{t_{i+1}}(t_i, x_i), \mathbf{x}_{i+2}), \\ &v_{i+1}^\pi(\mathbf{x}^i, X_{t_{i+1}}(t_i, x_i), \mathbf{x}_{i+2})) \Delta t_{i+1}, \\ V_i^\pi(\mathbf{x}^i, \omega, \mathbf{x}_{i+2}) &:= g(t_{i+1}, X_{t_{i+1}}(t_i, x_i), u_{i+1}^\pi(\mathbf{x}^i, X_{t_{i+1}}(t_i, x_i), \mathbf{x}_{i+2}), \\ &v_{i+1}^\pi(\mathbf{x}^i, X_{t_{i+1}}(t_i, x_i), \mathbf{x}_{i+2})), \end{aligned} \tag{4.2}$$

$$u_i^\pi(\mathbf{x}^i; \mathbf{x}_{i+1}) := \mathbb{E} U_i^\pi(\mathbf{x}^i, \omega, \mathbf{x}_{i+1}) + x_{i+1} \mathbb{E} V_i^\pi(\mathbf{x}^i, \omega, \mathbf{x}_{i+1}), \tag{4.3}$$

$$\begin{aligned} v_i^\pi(\mathbf{x}^i; \mathbf{x}_{i+1}) &:= \frac{1}{\Delta t_{i+1}} \mathbb{E} (U_i^\pi(\mathbf{x}^i, \omega, \mathbf{x}_{i+1}) \Delta W_{t_{i+1}}) \\ &+ \frac{x_{i+1}}{\Delta t_{i+1}} \mathbb{E} (V_i^\pi(\mathbf{x}^i, \omega, \mathbf{x}_{i+1}) \Delta W_{t_{i+1}}). \end{aligned} \tag{4.4}$$

**Theorem 4.1.** *We have for all  $i = 0, \dots, n$*

$$Y_{t_i}^\pi = u_i^\pi(X_{t_0}^\pi, \dots, X_{t_i}^\pi; \Delta B_{t_{i+1}}, \dots, \Delta B_{t_n}), \quad (4.5)$$

$$Z_{t_i}^{\pi,1} = v_i^\pi(X_{t_0}^\pi, \dots, X_{t_i}^\pi; \Delta B_{t_{i+1}}, \dots, \Delta B_{t_n}). \quad (4.6)$$

*Proof.* We proceed by backward induction. For  $i = n$ , by definition  $Y_{t_n}^\pi = \phi(X_{t_n}^\pi)$ , so (4.5) and (4.6) hold trivially.

Suppose that the result is true for  $j = n, n-1, \dots, i$ . The scheme described in (3.1) implies that

$$Y_{t_{i-1}}^\pi = Y_{t_i}^\pi + f(t_i, \Theta_{t_i}^{\pi,1}) \Delta t_i + g(t_i, \Theta_{t_i}^{\pi,1}) \Delta B_{t_i} - \int_{t_{i-1}}^{t_i} Z_r^\pi dW_r. \quad (4.7)$$

To prove (4.5), we take the conditional expectation of (4.7) with respect to  $\widehat{\mathcal{F}}_{t_{i-1}} = \mathcal{F}_{0,t_{i-1}}^W \vee \mathcal{F}_{0,T}^B$ ; this yields

$$\begin{aligned} \mathbb{E}(Y_{t_{i-1}}^\pi | \widehat{\mathcal{F}}_{t_{i-1}}) &= \mathbb{E}(Y_{t_i}^\pi | \widehat{\mathcal{F}}_{t_{i-1}}) + \mathbb{E}(f(t_i, \Theta_{t_i}^{\pi,1}) \Delta t_i | \widehat{\mathcal{F}}_{t_{i-1}}) \\ &\quad + \mathbb{E}(g(t_i, \Theta_{t_i}^{\pi,1}) \Delta B_{t_i} | \widehat{\mathcal{F}}_{t_{i-1}}) - \mathbb{E}\left(\int_{t_{i-1}}^{t_i} Z_r^\pi dW_r | \widehat{\mathcal{F}}_{t_{i-1}}\right). \end{aligned}$$

Using the fact that  $\int_{t_{i-1}}^{t_i} Z_r^\pi dW_r$  is orthogonal to any  $\widehat{\mathcal{F}}_{t_{i-1}}$ -measurable random variable, and the induction hypothesis we deduce:

$$\begin{aligned} Y_{t_{i-1}}^\pi &= \mathbb{E}(Y_{t_i}^\pi | \widehat{\mathcal{F}}_{t_{i-1}}) + \mathbb{E}(f(t_i, \Theta_{t_i}^{\pi,1}) \Delta t_i | \widehat{\mathcal{F}}_{t_{i-1}}) + \mathbb{E}(g(t_i, \Theta_{t_i}^{\pi,1}) \Delta B_{t_i} | \widehat{\mathcal{F}}_{t_{i-1}}) \\ &= \mathbb{E}\left(u_i^\pi(X_{t_0}^\pi, \dots, X_{t_{i-1}}^\pi, X_{t_i}(t_{i-1}, X_{t_{i-1}}^\pi), \Delta B_{t_{i+1}}, \dots, \Delta B_{t_n}) | \widehat{\mathcal{F}}_{t_{i-1}}\right) \\ &\quad + \Delta t_i \mathbb{E}\left(f(t_i, X_{t_i}(t_{i-1}, X_{t_{i-1}}^\pi), u_i^\pi(X_{t_0}^\pi, \dots, X_{t_{i-1}}^\pi, X_{t_i}(t_{i-1}, X_{t_{i-1}}^\pi), \Delta B_{t_{i+1}}, \dots, \Delta B_{t_n})) \right. \\ &\quad \left. v_i^\pi(X_{t_0}^\pi, \dots, X_{t_{i-1}}^\pi, X_{t_i}(t_{i-1}, X_{t_{i-1}}^\pi), \Delta B_{t_{i+1}}, \dots, \Delta B_{t_n}) | \widehat{\mathcal{F}}_{t_{i-1}}\right) \\ &\quad + \Delta B_i \mathbb{E}\left(g(t_i, X_{t_i}(t_{i-1}, X_{t_{i-1}}^\pi), u_i^\pi(X_{t_0}^\pi, \dots, X_{t_{i-1}}^\pi, X_{t_i}(t_{i-1}, X_{t_{i-1}}^\pi), \Delta B_{t_{i+1}}, \dots, \Delta B_{t_n})) \right. \\ &\quad \left. v_i^\pi(X_{t_0}^\pi, \dots, X_{t_{i-1}}^\pi, X_{t_i}(t_{i-1}, X_{t_{i-1}}^\pi), \Delta B_{t_{i+1}}, \dots, \Delta B_{t_n}) | \widehat{\mathcal{F}}_{t_{i-1}}\right). \end{aligned}$$

Since all  $\Delta B_{t_l}$ ,  $l = 1, \dots, n$  and  $X_{t_k}^\pi$ ,  $k = 0, \dots, i-1$  are  $\widehat{\mathcal{F}}_{t_{i-1}}$  measurable while  $W_{t_i} - W_{t_{i-1}}$  is independent of  $\widehat{\mathcal{F}}_{t_{i-1}}$ ; we deduce (4.5).

To prove (4.6), multiply (4.7) by  $\Delta W_{t_i} = W_{t_i} - W_{t_{i-1}}$  and take the conditional expectation with respect to  $\widehat{\mathcal{F}}_{t_{i-1}}$ , this yields

$$\begin{aligned} \mathbb{E}(Y_{t_{i-1}}^\pi \Delta W_{t_i} | \widehat{\mathcal{F}}_{t_{i-1}}) &= \mathbb{E}(Y_{t_i}^\pi \Delta W_{t_i} | \widehat{\mathcal{F}}_{t_{i-1}}) + \mathbb{E}(f(t_i, \Theta_{t_i}^{\pi,1}) \Delta t_i \Delta W_{t_i} | \widehat{\mathcal{F}}_{t_{i-1}}) \\ &\quad + \mathbb{E}(g(t_i, \Theta_{t_i}^{\pi,1}) \Delta B_{t_i} \Delta W_{t_i} | \widehat{\mathcal{F}}_{t_{i-1}}) - \mathbb{E}\left(\Delta W_{t_i} \int_{t_{i-1}}^{t_i} Z_r^\pi dW_r | \widehat{\mathcal{F}}_{t_{i-1}}\right). \end{aligned}$$

Since  $Y_{t_{i-1}}^\pi \in \widehat{\mathcal{F}}_{t_{i-1}}$  and  $\Delta W_{t_i}$  is independent of  $\widehat{\mathcal{F}}_{t_{i-1}}$  and centered we deduce

$$\mathbb{E}(Y_{t_{i-1}}^\pi \Delta W_{t_i} | \widehat{\mathcal{F}}_{t_{i-1}}) = 0.$$

Furthermore,

$$\begin{aligned} \mathbb{E}\left(\Delta W_{t_i} \int_{t_{i-1}}^{t_i} Z_r^\pi dW_r | \widehat{\mathcal{F}}_{t_{i-1}}\right) &= \mathbb{E}\left(\int_{t_{i-1}}^{t_i} Z_r^\pi dr | \widehat{\mathcal{F}}_{t_{i-1}}\right) \\ &= \mathbb{E}\left(\int_{t_{i-1}}^{t_i} Z_r^\pi dr | \mathcal{F}_{t_{i-1}}\right) = \Delta t_i Z_{t_i}^{\pi,1}. \end{aligned}$$

this completes the proof of (4.6).  $\square$

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